International Journal of Theoretical Physics, Vol. 5, No. 6 (1972), pp. 371–375

A Static Body of Arbitrarily Large Density

W. B. BONNOR and S. B. P. WICKRAMASURIYA

Queen Elizabeth College, (University of London), Atkins Building, Campden Hill, London W8 7AH

Received: 21 *June* 1971

Abstract

We consider the static, spherically symmetric, perfect fluid solution for a given mass m in general relativity, and show that its average density is bounded. We then show by an example that this need not be so for non-spherical bodies.

1. *Introduction*

In Newtonian mechanics we obtain a model of a spherical particle by taking a uniform sphere and allowing the radius r_0 to tend to zero, keeping the mass m constant. In this process the density (= mass/volume) tends to infinity.

The spherically symmetric interior solution for a perfect fluid in general relativity does not admit an analogous procedure. Using a usual notation we write

$$ds^2 = e^{\nu(r)}\,dt^2 - e^{\lambda(r)}\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

and assume that the pressure and the energy density ($= T_4{}^4$) are positive. Then to avoid singularities one finds (Bondi, 1964)

$$0 < e^{-\lambda} \leqslant 1,$$

so that the proper volume of the sphere is

$$V = 4\pi \int_0^{r_0} e^{\lambda/2}\, r^2\, dr \geqslant \frac{4\pi r_0{}^3}{3}$$

and the average active gravitational mass density is

$$\bar{\rho} = \frac{m}{V} \leqslant \frac{3m}{4\pi r_0{}^3}$$

Copyright © 1972 Plenum Publishing Company Limited. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of Plenum Publishing Company Limited.

It can also be shown (Bondi, 1964) that

$$\frac{m}{r_0} \leqslant 6\sqrt{(2)} - 8$$

hence

$$\bar{\rho} \leqslant \frac{6(3\sqrt{(2)} - 4)^3}{\pi m^2}$$

Hence given m the average density cannot be increased indefinitely by diminishing r_0.

Is a corresponding result also true for non-spherical bodies in general relativity? If not, we have another reason for looking upon the spherically symmetric solution with reserve, and for being cautious in using it on real bodies.

We put forward here a partial answer to the question, by showing that in at least one type of non-spherical body the density *can* be increased indefinitely.

2. *Electrically Counterpoised Dust* (*ECD*)

We construct our solution out of dust carrying electric charge of such density that the gravitation and electric repulsion just balance everywhere, and we call this material electrically counterpoised dust (ECD). The classical equilibrium solutions for this material were shown long ago (Majumdar, 1947; Papapetrou, 1947) to have strict and exact general relativistic analogues. The relativistic solution is the only static one known to us which need have no spatial symmetry whatever.

The metric is

$$ds^2 = -U^2(dx^2 + dy^2 + dz^2) + U^{-2}\,dt^2 \tag{2.1}$$

where U is a function of x, y, z. The Einstein–Maxwell equations

$$R^{ik} - \tfrac{1}{2}g^{ik}R = -8\pi(T^{ik} + E^{ik}) \tag{2.2}$$

$$4\pi E_k{}^i = -F^{ia}F_{ka} + \tfrac{1}{4}\delta_k{}^i F^{ab}F_{ab} \tag{2.3}$$

$$F_{ik} = A_{i,k} - A_{k,i} \tag{2.4}$$

$$F^{ik}{}_{;k} = 4\pi J^i \tag{2.5}$$

are satisfied by

$$T^{ik} = \rho v^i v^k, \qquad J^i = \sigma v^i \tag{2.6}$$

$$v^i = \delta_4{}^i U, \qquad A_i = \delta_i{}^4 \phi \tag{2.7}$$

$$\delta^{\alpha\beta} U_{,\alpha\beta} = -4\pi U^3 \rho \tag{2.8}$$

$$\phi = -\epsilon U^{-1} \tag{2.9}$$

$$\sigma = \epsilon\rho \tag{2.10}$$

where $\epsilon = \pm 1$, $\alpha, \beta = 1, 2, 3$, comma means partial differentiation and semi-colon covariant differentiation. The units have been chosen so that

$c=1$, $G=1$, so, as shown in (2.9) and (2.10), the gravitational and electric potentials are equal in magnitude, and so are the mass and charge densities.

Equation (2.8) presents us with a static potential problem of classical type. Consider a single isolated ECD body: outside it $\sigma=\rho=0$ so (2.8) gives

$$\delta^{\alpha\beta}\underset{\mathrm{E}}{U}{}_{,\alpha\beta}=0 \tag{2.11}$$

$\underset{\mathrm{E}}{U}$ referring to the exterior. For the interior function $\underset{\mathrm{I}}{U}$, (2.8)–(2.10) apply and at the boundary B sufficient matching conditions are

$$\underset{\mathrm{I}}{U}=\underset{\mathrm{E}}{U} \qquad \underset{\mathrm{I}}{U}{}_{,\alpha}=\underset{\mathrm{E}}{U}{}_{,\alpha} \tag{2.12}$$

Now let V denote the proper volume of the body, and V' that of its Euclidean image, viz:

$$V=\iiint \underset{\mathrm{I}}{U}{}^{3}\,dx\,dy\,dz, \qquad V'=\iiint dx\,dy\,dz \tag{2.13}$$

Then we have

$$\underset{\mathrm{I}}{U}{}^{3}_{\min}V'<V<\underset{\mathrm{I}}{U}{}^{3}_{\max}V' \tag{2.14}$$

where $\underset{\mathrm{I}}{U}{}_{\min}$, $\underset{\mathrm{I}}{U}{}_{\max}$ are the least and greatest values of U within V and on B. We notice that, taking U positive (with no loss of generality) (2.8) tells us that provided $\rho>0$ U cannot have a true minimum inside V so

$$\underset{\mathrm{I}}{U}{}_{\min}=\text{least value of } U \text{ on } B \overset{\text{def}}{=} U_B$$

(2.14) gives for the average density the limits

$$\frac{m}{\underset{\mathrm{I}}{U}{}^{3}_{\max}V'}<\bar{\rho}<\frac{m}{\underset{\mathrm{B}}{U}{}^{3}V'} \tag{2.15}$$

If we allow V' to tend to zero there are in (2.15) two possibilities of interest:

(i) $U_B{}^3V'\to$ finite limit so $\bar{\rho}$ is bounded. Since the limit is independent of $\underset{\mathrm{I}}{U}$, the bound for $\bar{\rho}$ is independent of the density distribution inside the body.

(ii) $\underset{\mathrm{I}}{U}{}^{3}_{\max}V'\to$ zero so $\bar{\rho}\to\infty$.

3. *Two Examples*

We give an example of each sort of behaviour.

(i) Consider

$$\underset{\mathrm{E}}{U}=1+\frac{m}{r}, \qquad r\geqslant r_0 \tag{3.1}$$

$$\underset{\mathrm{I}}{U}=1+\frac{m}{r_0}\left(1+\frac{r_0{}^4-r^4}{4r_0{}^4}\right), \qquad 0\leqslant r<r_0 \tag{3.2}$$

where $r = +(x^2 + y^2 + z^2)^{1/2}$. This solution represents a spherically symmetric and non-singular distribution of ECD. Straightforward calculation gives for (2.15)

$$\frac{3m}{4\pi(r_0 + \frac{5}{4}m)^3} < \bar{\rho} < \frac{3m}{4\pi(r_0 + m)^3} \tag{3.3}$$

so, given m, $\bar{\rho}$ is bounded above, no matter how small r_0 may be. As remarked at the end of Section 2, the upper bound for $\bar{\rho}$ is independent of $\underset{\mathrm{I}}{U}$. This case is similar to the spherically symmetric solution for a perfect fluid.

(ii) In our second example we use oblate spheroidal coordinates, taking for the metric

$$ds^2 = -U^2[a^2(\sinh^2 u + \sin^2\theta)(du^2 + d\theta^2) + a^2\cosh^2 u \cos^2\theta \, d\phi^2] + U^{-2}\, dt^2 \tag{3.4}$$

$$\underset{\mathrm{E}}{U} = 1 + \frac{m}{a}\tan^{-1}(\operatorname{cosech} u), \quad u \geqslant u_0 > 0, \quad 0 \leqslant \tan^{-1}(\operatorname{cosech} u) \leqslant \frac{\pi}{2} \tag{3.5}$$

$$\underset{\mathrm{I}}{U} = 1 + \frac{m}{a}\left[\tan^{-1}(\operatorname{cosech} u_0) + \frac{u_0{}^4 - u^4}{4u_0{}^3}\operatorname{sech} u_0\right], \qquad 0 \leqslant u < u_0 \tag{3.6}$$

the ranges of θ, ϕ and t are

$$-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}, \qquad 0 \leqslant \phi \leqslant 2\pi, \qquad -\infty < t < \infty \tag{3.7}$$

and a, u_0 are constants. $\underset{\mathrm{E}}{U}$ satisfies (2.11), and (2.12) is also satisfied. The solution is non-singular and represents the interior and exterior field of an oblate spheroid of ECD, of mass m. (For a fuller discussion of spheroidal metrics see Bonnor & Sackfield (1968).)

We find, using (3.4) and (3.7), that

$$V' = \frac{4\pi a^3}{3}\sinh u_0 \cosh^2 u_0 \tag{3.8}$$

The inequality (2.15) gives

$$\frac{3m \operatorname{cosech} u_0 \operatorname{sech}^2 u_0}{4\pi a^3\left\{1 + \frac{m}{a}\left[\tan^{-1}(\operatorname{cosech} u_0) + \frac{u_0}{4}\operatorname{sech} u_0\right]\right\}^3} < \bar{\rho} < \frac{3m \operatorname{cosech} u_0 \operatorname{sech}^2 u_0}{4\pi a^3\left\{1 + \frac{m}{a}[\tan^{-1}(\operatorname{cosech} u_0)]\right\}^3} \tag{3.9}$$

and we see that $\bar{\rho}$ can become arbitrarily large if u_0 is allowed to tend to zero. *This means that as the spheroid tends towards a disc its average gravitational mass density tends to infinity.*

References

Bondi, H. (1964). *Proceedings of the Royal Society* A, **282**, 303.
Bonnor, W. and Sackfield, A. (1968). *Communications in Mathematical Physics*, **8**, 338.
Majumdar, S. D. (1947). *Physical Review*, **72**, 390.
Papapetrou, A. (1947). *Proceedings of the Royal Irish Academy*, **51**, 191.